On unconstrained $SU(2)$ gluodynamics with theta angle

A.M. Khvedelidze[1,2], D.M. Mladenov[1], H.-P. Pavel[1,3], G. Röpke[3]

[1] Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia
[2] Department of Theoretical Physics, A. Razmadze Mathematical Institute, 380093 Tbilisi, Georgia
[3] Fachbereich Physik der Universität Rostock, 18051 Rostock, Germany

Abstract. The Hamiltonian reduction of classical $SU(2)$ Yang–Mills field theory to the equivalent unconstrained theory of gauge invariant local dynamical variables is generalized to the case of nonvanishing θ-angle. It is shown that for any θ-angle the elimination of the pure gauge degrees of freedom leads to a corresponding unconstrained non-local theory of self-interacting second rank symmetric tensor fields, and that the obtained classical unconstrained gluodynamics with different θ-angles are canonically equivalent as on the original constrained level.

1 Introduction

The gauge and Poincaré invariant action of Yang–Mills theory depends on two parameters, the coupling constant g and the so-called θ-angle, as coefficients in front of the CP even part $S^{(+)}$,

$$S^{(+)} = \frac{1}{2g^2} \int \mathrm{d}^4 x \mathrm{tr} F_{\mu\nu} F^{\mu\nu}, \qquad (1)$$

and the CP odd part $S^{(-)}$,

$$S^{(-)} = -\frac{\theta}{16\pi^2} \int \mathrm{d}^4 x \mathrm{tr}^* F_{\mu\nu} F^{\mu\nu}, \qquad (2)$$

respectively. At the classical level neither the value of the coupling constant nor that of the θ-angle effect the observables, because the complete information for the description of the classical behaviour of the gauge fields is coded entirely in the extremum of the action. When all components of the gauge potential entering the action are varied as independent variables the topological charge density term $Q(x) = -(1/16\pi^2)\mathrm{tr}^* F_{\mu\nu} F^{\mu\nu}$ can be discarded as a total divergence,

$$Q(x) = \partial_\mu K^\mu, \qquad (3)$$

with the Chern–Simons current K^μ [1]

$$K^\mu = -\frac{1}{16\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\mathrm{tr}\left(F_{\alpha\beta} A_\gamma - \frac{2}{3} A_\alpha A_\beta A_\gamma\right), \qquad (4)$$

and thus the extremal curves are independent of both the coupling constant and the θ-angle.

Passing to the quantum theory it is generally believed [2–4] that the physical observables become θ-dependent. Although in perturbative calculations all diagrams with vertex $Q(x)$ vanish, non-perturbative phenomena such as

tunneling between the above topologically distinct classical vacua, labeled by the integer value of the winding number functional

$$W[A] = \int \mathrm{d}^3 x K^0, \qquad (5)$$

leads to the appearance of θ-vacua. Configurations with different winding numbers are related to each other by large gauge transformations reflecting the fact that the topological current K_μ is not gauge invariant.

We therefore ask at this place the question whether it is possible to express the topological term in the classical action as a total divergence of a gauge invariant current using the unconstrained formulation of gauge theories [5–20]. In the hope to obtain such a representation of the topological term we would like to generalize in the present note the Hamiltonian reduction of classical $SU(2)$ Yang–Mills field theory given in [18] to arbitrary θ-angle by including the CP odd part (2) of the action. We shall reformulate the original degenerate Yang–Mills theory as an unconstrained non-local theory of self-interacting second rank symmetric tensor fields.

Carrying out such a reduction in the presence of a total divergence term in the action one can meet the so-called *"divergence problem"* specific for the field theory with constraints which has no analog for finite-dimensional mechanical systems. This problem has first been formulated explicitly in the context of the canonical reduction of general relativity[1]. Forty years ago Arnowitt, Deser and Mis-

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[1] Presumably, the idea of the importance of the careful consideration of terms which are total spatial divergences goes back to Dirac in 1959 when he constructed the reduced Hamiltonian in general relativity as a certain surface integral at spatial infinity [21]

ner [22] gave a clear and vivid formulation of the phenomenon: *"a term which in the original Lagrangian (or Hamiltonian) is a pure divergence, may cease to be a divergence upon elimination of the redundant variables and hence may contribute to the equations of motion obtained from the reduced Lagrangian (Hamiltonian)"*. A simple ad hoc example from [22] explains the idea of this statement. Consider a theory where among the variables there is a redundant variable satisfying the constraint

$$\nabla^2 \Phi = \chi^2. \tag{6}$$

A term $\nabla^2 \Phi$ added to the degenerate Lagrangian being a divergence has no influence on the classical equation of motion, while after projection onto the constraint shell it appears as χ^2 and would contribute to the equations of motion.

We shall demonstrate that the Hamiltonian reduction of $SU(2)$ Yang–Mills gauge theory is free of the above mentioned divergence problem due to the Bianchi identities. Equivalence of constrained and unconstrained formulations of gauge theories on the classical level requires the demonstration of the agreement between reduced and original non-Abelian Lagrangian equations of motion. We shall explicitly construct the canonical transformation, well defined on the reduced phase space, that eliminates the θ-dependence of the classical equations of motion for the unconstrained variables[2].

2 Theta independence on the constrained level

Let us first review the case of the original constrained theory and demonstrate that under the special boundary conditions for the fields at spatial infinity (see (15) below) there exists a canonical transformation which completely eliminates the θ-dependence from the classical degenerate theory.

2.1 Hamiltonian formulation of the constrained theory

Both parts of the action $S = S^{(+)} + S^{(-)}$ are invariant under the local gauge transformations

$$A_\mu \to A'_\mu = U^{-1}(x)\,(A_\mu - \partial_\mu)\,U(x), \tag{7}$$

with an arbitrary space-time depended element $U(x)$ of the gauge group. This means that the Lagrangian theory is degenerate and the standard Hamiltonian description needs to be generalized. We shall follow the Dirac generalized Hamiltonian approach [24, 25].

Inclusion of the CP odd part of the action $S^{(-)}$ leads to the modification of the canonical momenta

$$\Pi_a = \frac{\partial L}{\partial \dot{A}_{a0}} = 0, \tag{8}$$

$$\Pi_{ai} = \frac{\partial L}{\partial \dot{A}_{ai}} = \frac{1}{g^2}\left(\dot{A}_{ai} - (D_i(A))_{ac}A_{c0}\right) + \frac{\theta}{8\pi^2}B_{ai}, \tag{9}$$

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[2] A similar construction for gravity has been done recently in [23]

where the covariant derivative D_i reads

$$(D_i(A))_{mn} = \delta_{mn}\partial_i + (J^c)_{mn}\,A_{ci}, \tag{10}$$

with the 3×3 matrix generators of the $SO(3)$ group $(J_s)_{mn} := \epsilon_{msn}$, and non-Abelian magnetic fields

$$B_{ai} = \varepsilon_{ijk}\left(\partial_j A_{ak} + \frac{1}{2}\epsilon_{abc}A_{bj}A_{ck}\right) \tag{11}$$

have been introduced. Independently of this modification the phase space spanned by the variables (A_{a0}, Π_a) and (A_{ai}, Π_{ai}) is restricted by the three primary constraints $\Pi_a(x) = 0$.

The canonical Hamiltonian is

$$H_C = \int \mathrm{d}^3 x \left[\frac{g^2}{2}\left(\Pi_{ai} - \frac{\theta}{8\pi^2}B_{ai}\right)^2 \right.$$
$$\left. + \frac{1}{2g^2}B_{ai}^2 + \Pi_{ai}\,(D_i A_0)_a\right], \tag{12}$$

where we have used that the topological charge density $Q(x)$ can be rewritten in terms of the non-Abelian electric and magnetic fields as

$$Q = \frac{1}{8\pi^2}F_{0i}^a B_{ai}. \tag{13}$$

The standard way in the Hamiltonian approach to proceed further, is to perform a partial integration in the last term in expression (12) for the canonical Hamiltonian

$$\int_{V_R} \mathrm{d}^3 x \Pi_{ai}\,(D_i A_0)_a = -\int_{V_R} \mathrm{d}^3 x A_{a0}\,(D_i \Pi_i)_a$$
$$+ \oint_{\Sigma_R} \mathrm{d}^2\sigma_i A_{a0}\Pi_{ai}, \tag{14}$$

where according to the Gauss theorem the surface integral is over the two-dimensional closed surface covering the three-dimensional volume V_R (for simplicity we assume that it is a ball with radius R). Supposing that

$$\lim_{R\to\infty} \oint_{\Sigma_R} \mathrm{d}^2\sigma_i A_{a0}\Pi_{ai} = 0, \tag{15}$$

we obtain the non-Abelian Gauss law constraint

$$(D_i)_{ac}\,\Pi_{ic} = 0, \tag{16}$$

as the condition to maintain the primary constraints $\Pi_a = 0$ during the evolution. According to the Dirac prescription the generator of the time translation is the total Hamiltonian

$$H_T = \int \mathrm{d}^3 x \left[\frac{g^2}{2}\left(\Pi_{ai} - \frac{\theta}{8\pi^2}B_{ai}\right)^2 + \frac{1}{2g^2}B_{ai}^2 \right.$$
$$\left. - A_{a0}D_i\Pi_{ai} + \lambda_a \Pi_a\right], \tag{17}$$

depending on three arbitrary functions $\lambda_a(x)$, and the Poisson brackets have a canonical structure

$$\{A_{ai}(\boldsymbol{x}, t), \Pi_{bj}(\boldsymbol{y}, t)\} = \delta_{ab}\delta_{ij}\delta^3(\boldsymbol{x} - \boldsymbol{y}), \tag{18}$$

$$\{A_{a0}(\boldsymbol{x}, t), \Pi_b(\boldsymbol{y}, t)\} = \delta_{ab}\delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{19}$$

2.2 Canonical equivalence of constrained theories with different theta angles

Based on the representation (17) for the total Hamiltonian one can immediately verify the equivalence of classical theories with different values of the parameter θ. To convince ourselves let us perform the transformation to the new coordinates A_{ai} and E_{bj}:

$$A_{ai}(x) \rightarrow A_{ai}(x) = A_{ai}(x), \qquad (20)$$

$$\Pi_{bj}(x) \rightarrow E_{bj} = \Pi_{bj}(x) - \frac{\theta}{8\pi^2} B_{bj}(x). \qquad (21)$$

One can easily check that this transformation is canonical; the new coordinates A_{ai} and E_{ai} satisfy the same canonical Poisson brackets relations (18) as the original one. And noticing that by virtue of the Bianchi identity

$$\epsilon^{\mu\nu\lambda\rho} D_\nu F_{\lambda\rho} = 0, \qquad (22)$$

one can conclude that the θ-dependence completely disappears from the Hamiltonian (17).

Note that the canonical transformation (21) can be represented in the form

$$E_{ai} = \Pi_{ai} - \theta \frac{\delta}{\delta A_{ai}} W[A], \qquad (23)$$

where $W[A]$ denotes the winding number functional (5).

3 Theta independence on the unconstrained level

We shall now derive the unconstrained version of Yang–Mills theory with θ-angle and then give the analog of the transformation (20) and (21) after projection to the reduced phase space, thus checking the consistency of the unconstrained canonical formulation of Yang–Mills theory.

3.1 Hamiltonian formulation of the unconstrained theory

For the reduction of $SU(2)$ Yang–Mills theory we shall follow the method developed in [18] for the CP even part of the action. To reduce the CP odd part one can proceed similarly.

Let us therefore perform the following point transformation to the new set of Lagrangian coordinates q_j ($j = 1, 2, 3$) and the six elements $S_{ik} = S_{ki}$ ($i, k = 1, 2, 3$) of the positive definite symmetric 3×3 matrix S:

$$A_{ai}(q, S) = O_{ak}(q) S_{ki} - \frac{1}{2} \epsilon_{abc} \left(O(q) \partial_i O^{\mathrm{T}}(q) \right)_{bc}, \quad (24)$$

where $O(q)$ is an orthogonal 3×3 matrix parameterized by the three fields q_i.

The first term in (24) corresponds to the so-called polar decomposition for arbitrary quadratic matrices. The inclusion of the additional second term is motivated by the inhomogeneity of the gauge transformation (7)[3]. The transformation (24) induces a point canonical transformation linear in the new conjugated momenta P_{ik} and p_i. Using the corresponding generating functional depending on the old momenta and the new coordinates,

$$F_3[\Pi; q, S] = \int \mathrm{d}^3 z \Pi_{ai}(z) A_{ai}(q(z), S(z)), \qquad (25)$$

one can obtain the transformation to new canonical momenta p_i and P_{ik}

$$p_j(x) = \frac{\delta F_3}{\delta q_j(x)} = -\Omega_{jr} \left(D_i(Q) S^{\mathrm{T}} \Pi \right)_{ri}, \qquad (26)$$

$$P_{ik}(x) = \frac{\delta F_3}{\delta S_{ik}(x)} = \frac{1}{2} \left(\Pi^{\mathrm{T}} O + O^{\mathrm{T}} \Pi \right)_{ik}. \qquad (27)$$

Here

$$\Omega_{ji}(q) := -\frac{1}{2} \mathrm{Tr} \left(O^{\mathrm{T}}(q) \frac{\partial O(q)}{\partial q_j} J_i \right). \qquad (28)$$

The symplectic structure of the new variables is encoded in the fundamental Poisson brackets[4]

$$\{ S_{ij}(x), P_{kl}(y) \} = \frac{1}{2} \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \delta^{(3)}(x - y). \quad (29)$$

A straightforward calculation based on the linear relations (26) and (27) between the old and the new momenta leads to the following expression for old momenta Π_{ai} in terms of the new canonical variables:

$$\Pi_{ai} = O_{ak}(q) \left[P_{ki} + \epsilon_{kis} P_s \right], \qquad (30)$$

where the vector P_s is a solution to the system of first order partial differential equations

$$^* D_{ks}(S) P_s = s_k(x) + \Omega_{kl}^{-1} p_l. \qquad (31)$$

In (31) the $^*D$ denotes the matrix operator

$$^* D_{ik}(S) = -(D_m(S) J^m)_{ki}, \qquad (32)$$

and one can verify that the vector

$$s_k(x) = (D_i(S))_{kl} P_{il} \qquad (33)$$

coincides up to a divergence term with the spin density part of the Noetherian angular momentum calculated in terms of the new variables and projected onto the constraint shell. Using the representations (24) and (30) one can easily convince oneself that the new variables S and P make no contribution to the Gauss law constraints (16):

$$O_{as}(q) \Omega_{sj}^{-1}(q) p_j = 0. \qquad (34)$$

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[3] One can treat (24) as a gauge transformation to the new field configuration $S(x)$ which satisfy the so-called symmetric gauge condition $\epsilon_{abc} S_{bc} = 0$. The uniqueness and regularity of the transformation (24) depends on the boundary conditions imposed

[4] These new brackets take into account the symmetry constraints $S_{ij} = S_{ji}$ and $P_{kl} = P_{lk}$ and rigorously speaking are Dirac brackets

Here and in (31) we assume that the matrix Ω is invertible and thus the equivalent set of Abelian constraints is

$$p_a = 0. \tag{35}$$

The Abelian form of the Gauss law constraints is the main advantage of the new variables. In terms of these coordinates the projection to the constraints shell is achieved by vanishing values for the momenta p_a in all expressions.

The reduced Hamiltonian is defined as the projection of the total Hamiltonian to the constraint shell $p_a = 0$ and $\Pi_a = 0$. In terms of the unconstrained canonical variables S and P it reads

$$H = \int \mathrm{d}^3x \left[\frac{g^2}{2} \left(P_{ai} - \frac{\theta}{8\pi^2} B_{(ai)} \right)^2 \right.$$
$$\left. + g^2 \left(P_a - \frac{\theta}{8\pi^2} B_a \right)^2 + \frac{1}{2g^2} B_{ai}^2 \right]. \tag{36}$$

Here $B_{(ai)}$ and B_a denote the symmetric tensor $B_{(ai)} = (B_{ai} + B_{ia})/2$ and vector $B_a = \epsilon_{abc} B_{bc}/2$ constructed from the chromomagnetic field

$$B_{sk} = \epsilon_{klm} \left(\partial_l S_{sm} + \frac{1}{2} \epsilon_{sbc} S_{bl} S_{cm} \right). \tag{37}$$

The vector P_a representing the non-local term in the Hamiltonian (36) is given as the solution to the system of differential equations

$$^*D_{ks}(S) P_s = s_k(x), \tag{38}$$

which is the projection of (31) to the constraint surface $p_a = 0$.

3.2 Canonical equivalence of the unconstrained theory with different theta angles

For the original degenerate action in terms of the A_μ fields the equivalence of classical theories with arbitrary value of θ-angle has been reviewed in Sect. 2. Let us now examine the same problem for the derived unconstrained theory considering the analog of the canonical transformation (20) and (21) after projection onto the constraint surface

$$S_{ai}(x) \rightarrow S_{ai}(x) = S_{ai}(x), \tag{39}$$

$$P_{bj}(x) \rightarrow E_{bj}(x) = P_{bj}(x) - \frac{\theta}{8\pi^2} B_{(bj)}(x). \tag{40}$$

First of all one can easily check that this transformation to the new variables S_{ai} and E_{bj} is canonical with respect to the Dirac brackets (29). The Hamiltonian (36) in terms of the new variables S_{ai} and E_{bj} is therefore θ-independent. It looks like

$$H = \int \mathrm{d}^3x \left[\frac{g^2}{2} E_{ai}^2 + g^2 E_a^2 + \frac{1}{2g^2} B_{ai}^2 \right], \tag{41}$$

where E_a is a solution to (38) with the replacement $P_{ai} \rightarrow E_{ai}$. This follows from the observation that if P_a is a solution to (38) then the expression

$$E_a = P_a - \frac{\theta}{8\pi^2} B_a \tag{42}$$

is a solution to the same equation with the replacement $P_{ai} \rightarrow E_{ai}$. This is indeed valid because the B_{ai} field satisfies the identity

$$^*D_{ks}(S) B_s = (D_i(S))_{kl} B_{(li)}. \tag{43}$$

Equation (43) is the Bianchi identity $(D_i)_{ab} B_{bi} = 0$ rewritten in terms of the symmetric $B_{(ai)}$ and antisymmetric B_a parts of the chromomagnetic field strength.

The reduced form of the generating functional (5) corresponding to the transformation (40) is the same functional W evaluated for the symmetric tensor S_{ik}. One can convince oneself that the symmetric part of the magnetic field $B_{(ij)}(S)$ can be written as the functional derivative of this functional $W[S]$:

$$\frac{\delta}{\delta S_{ij}(x)} W[S] = \frac{1}{8\pi^2} B_{(ij)}(x), \tag{44}$$

and thus the canonical transformation that eliminates the θ-dependence from the Hamiltonian can be represented in the same form as (23) with the nine gauge fields A replaced by the six unconstrained fields $S_{ik}(x)$.

4 Concluding remarks

We have explored the question of the θ-independence of classical unconstrained $SU(2)$ gluodynamics in order to build the basis for passing to the quantum level. We have shown that the exact projection of $SU(2)$ gluodynamics to the reduced phase space leads to an unconstrained system whose classical equations of motion are consistent with the original degenerate theory in the sense that they are θ-independent. The crucial point is that the fulfillment of this condition is due to properly taking into account the Bianchi identity for the magnetic field. As a consequence of the independence of the classical equations of motion of the gauge invariant local fields, the parity odd term in the Yang–Mills action is a total divergence of some gauge invariant current, in contrast to the original unconstrained theory, where it was the total divergence of the gauge variant Chern–Simons current K_μ. The explicit construction of the gauge invariant current in the unconstrained theory remains a topic for further investigation. Furthermore, to deal practically with such a complicated non-local Hamiltonian as (36) one would have to use some approximation, because the exact solution to (38) is unknown. Implementing one or another approximate solution it is desirable to be consistent with the θ-independence of classical theory.

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